# A Survey of Linear Extremal Problems in Analytic Function Spaces

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Abstract. The purpose of this survey paper is to recall the major benchmarks of the theory of linear extremal problems in Hardy spaces and to outline the current status and open problems remaining in Bergman spaces. We focus on the model extremal problem of maximizing the norm of the linear functional associated with integration against a polynomial of finite degree, and discuss known solutions of particular cases of that problem. We examine duality and its application in both Hardy and Bergman spaces. Finally, we discuss some recent progress on the finiteness of the Blaschke product of the extremal solution in Bergman spaces.

## 1. Introduction and Historical Remarks

Solving extremal problems has been one of the major stimuli for progress in complex analysis, starting with the Schwarz lemma in the late 19th century, followed by work on coefficients of bounded analytic functions by C. Caratheodory and L. Fejer, Landau, Szasz, and others. At the end of the First World War, F. Riesz considered a best approximation problem in the Hardy space $H^1$, and in 1926, Szasz associated this problem with a dual problem in $H^{\infty}$. This duality was rediscovered by Geronimus and, in a more general framework, by Krein in 1938. Extremal problems in multiply connected domains were studied by Grunsky (1940), Heins (1940), Robinson (1943), Goluzin (1946), and Ahlfors (1947). Macintyre and Rogosinski (1950) gave a detailed survey of results related to extremal problems involving coefficients of functions in all Hardy classes. Systematic use of duality in linear extremal problems for analytic functions started with S. Ya. Khavinson (1949) and independently Rogosinski and Shapiro (1953). Further studies were undertaken by Bonsall, Royden, Read, Adamyan, Arov, Krein, Walsh, among others. For a full account of the history of the development of extremal problems and references, see [18, pp. 51–57].

Work on Bergman spaces began with Ryabych in the early 1960s, who started the investigation of the existence and regularity of solutions ([21, 22]). In 1991, Osipenko and Stessin ([19]) solved an explicit optimization problem in Bergman spaces involving linear combinations of the value of a function and its derivative

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at a particular point in the disk. The theory of contractive divisors in Bergman spaces, initiated by Hedenmalm ([7]), followed by Duren, D. Khavinson, Shapiro and Sundberg ([2, 3]), prompted a burst of activity in Bergman spaces which gave insight into the structure of the $z$-invariant subspaces of Bergman spaces. These developments are recorded in two books on Bergman spaces ([4, 8]). In 1997, D. Khavinson and Stessin made a deeper study of linear extremal problems in Bergman spaces ([9]). Ferguson gave a simpler proof of Ryabych's regularity results and generalized them in 2009 and 2010 ([5]). The paper [26] contains a nice discussion of results on extremal problems in Bergman spaces.

The purpose of this survey is to recall the major benchmarks of the theory in Hardy spaces and to outline the current status of developments in Bergman spaces as well as the obstacles that still remain there. The plan of the paper is as follows: we begin in Section 2 by defining Bergman spaces, state a model extremal problem, and investigate the existence and uniqueness of extremals. In Section 3, we give examples and known solutions of that extremal problem in special cases. Section 4 discusses the Duality Theorem, and in Section 5, we apply duality to see how to get the solutions in Hardy spaces. In Section 6, we tackle the Bergman space case, discuss the difficulties and examine the connection with partial differential equations. In Section 7, we give a proof of a new result that the Blaschke product of the extremal solution for Bergman spaces $A^p$ for $p$ close to 2 is finite.

## 2. A Model Extremal Problem in Bergman Spaces

Let us begin by examining a model extremal problem in the Bergman space.

**Definition 2.1.** For $0 < p < \infty$; define the Bergman space as

$$A^p = \left\{ f \text{ analytic in } \mathbb{D} : \left( \int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} =: \|f\|_{A^p} < \infty \right\};$$

where $dA(z) = \frac{1}{\pi} dxdy$ denotes normalized area measure in the unit disk $\mathbb{D}$; $z = x + iy$.

Consider the following model extremal problem: Fix $1 \le p < \infty$. Given a non-zero polynomial

$$\omega(z) = \sum_{k=0}^{N} a_k z^k;$$

describe the extremal solutions of the problem:

$$\lambda_p := \sup \left\{ \mathrm{Re} \int_{\mathbb{D}} f(z)\overline{\omega(z)}\, dA(z) : \|f\|_{A^p} \le 1 \right\} \tag{2.1}$$

or, equivalently,

$$\sup \left\{ \mathrm{Re} \left( \sum_{0}^{N} c_k f^{(k)}(0) \right); c_k = \frac{\overline{a_k}}{(k+1)!}, \|f\|_{A^p} \le 1 \right\}. \tag{2.2}$$

Solving Problem (2.1) is equivalent to solving the following problem:

$$\inf \left\{ \|F\|_{A^p} : \int_{\mathbb{D}} F\overline{\omega}\, dA = 1 \right\}; \tag{2.3}$$

since it is easily checked that $F$ is a solution to (2.3) if and only if $f$ is a solution to (2.1), where $F = f$ ==71174 0 l SQBT/F34 9.9626 T2ck7or 0

For $1 \le p < \infty$, if $\{F_n\}$ is a sequence of $A^p$ functions approaching the infimum in (2.3), then their $A^p$ norms are bounded, and thus, thinking of these functions as linear functionals on $A^q$ for $1/p + 1/q = 1$, by the weak* compactness of bounded sets in $A^p$, there exists a function $F \in A^p$ and a subsequence $F_{n_k}$ of $F_n$ such that $F_{n_k}$ approaches $F$ weak*. In particular,

$$1 = \int_{\mathbb{D}} F_{n_k} \overline{\omega}\, dA \to \int_{\mathbb{D}} F \overline{\omega}\, dA,$$

and therefore

$$\int_{\mathbb{D}} F \overline{\omega}\, dA = 1,$$

and of course, $F_{n_k} \to F$ pointwise. Finally, by Fatou's theorem, $\|F\|_{A^p} \le \liminf \|F_{n_k}\|_{A^p}$, and therefore

$$\|F\|_{A^p} = \inf \left\{ \|F\|_{A^p} : \int_{\mathbb{D}} F \overline{\omega}\, dA = 1 \right\},$$

as desired.

If $p = 1$, the argument is similar but slightly more delicate, since to use weak* compactness, we must think of $A^1$ as a subset of the set of complex measures on $\overline{\mathbb{D}}$. In this case, for a sequence $\{F_n\}$ of $A^1$ functions approaching the infimum in (2.3), the measures $F_n dA$ form a bounded sequence of measures on the disk, and therefore, by weak* compactness of bounded measures on $\overline{\mathbb{D}}$, there exists a measure $d\mu$ such that some subsequence $F_{n_k} dA$ approaches $d\mu$ weak*, that is

$$\int_{\mathbb{D}} F_{n_k} f\, dA \to \int_{\mathbb{D}} f\, d\mu,$$

for every $f$ continuous in $\overline{\mathbb{D}}$. We now appeal to a version of the F&M Riesz theorem for $A^1$ proved by H. Shapiro ([23, 24]), which can be stated as follows.

**Theorem 2.2.** Let $\Omega$ be any bounded open se[illegible] t le $M(\overline{\Omega})$ be [illegible] t [illegible] $\overline{\Omega}$, and suppose $f_n \in A^1(\Omega)$ is a se[illegible] $f_n dA \to d\mu$ weak*, for some $\mu \in M(\overline{\Omega})$. Then [illegible] $f \in A^1(\Omega)$ such [illegible] $d\mu = f dA$. [illegible]

Here, $A^1(\Omega)$ is naturally defined as the space of integrable analytic functions in the domain $\Omega$. Note that in the original statement of this theorem, the domain $\Omega$ is allowed to have non-smooth boundary points, and then the limit measure is of the form $f d\mu + d\nu$, where $\nu$ a singular measure supported on these non smooth boundary points. See pp. 75–76 of [24] for details.

Now, getting back to the proof of existence, we see that by Theorem 2.2 applied to $\mathbb{D}$, the measure $\mu$ in question is absolutely continuous, and therefore, there exists a function $F$ such that the measures $F_{n_k} dA$ approach $F\, dA$ weak*. The rest of the argument is the same as for $p > 1$, since $\omega$ is continuous.

Finally, the Bergman spaces $A^p$ are strictly convex for all $1 \le p < \infty$ (see, for example, [4, pp. 28–29]), which implies that there can only be one element of minimal norm satisfying $\int_{\mathbb{D}} F \overline{\omega}\, dA = 1$. Therefore the solution to Problem 2.3 and therefore to Problem 2.1 is unique. Note that for $1 < p < \infty$, the argument showing existence and uniqueness of an extremal for the model problem considered here immediately extends to $\omega \in A^q$, for $1/p + 1/q = 1$.

The main thrust of this work is to study the smoothness properties of the extremal functions. It is always expected that the solution of a "nice" extremal

*Example* 3.5. Problems of the type considered in Examples 1 through 4 are connected to what are often called Caratheodory-Fejer type problems. An important example is the problem of nding, for given $| \;_j| < 1$ for $j = 1, \ldots, m$:

$$\inf\{\|f\|_{A^p} : f^{(N)}(0) = 1; f(0) = \quad = f^{(N} \tag{3.1}$$

product to $\mathbb{C}\setminus\{0\}$: Moreover, at $z = 0$, the singularity is given by the behavior of $\bar{f}$; and hence is a pole of order $N$; and therefore by reflection a similar behavior occurs at

where $u \in W_0^{1;q}$. Now set $v(z) := u(z) + \overline{\ (z)}$; $(z) := \int_0^z \omega(\ )d\ $; so that

$$\frac{\partial v}{\partial z} = \frac{|f|^p}{f}.$$

Then $v$ solves the nonlinear boundary value problem:

$$\frac{\partial}{\partial z}(|v_{\bar z}|^{q\ 2} v_{\bar z}) = 0 \text{ in } \mathbb{D}; \qquad (6.4)$$

$$v = \overline{\ } \text{ on } \mathbb{T}.$$

By results of Ch. Morrey, O. Ladyzhenskaya, and N. Uraltseva (see the discussion in [9] and [13, 14, 15, 10 ]), the unique solution $v$ of (6.4) belongs to $C^{1+}\ (\overline{\mathbb{D}})$; $=$ $(q)$. Since

$$f = {}^{q\ 1}\frac{|v_{\bar z}|^q}{v_{\bar z}};$$

we get that $f \in \mathrm{Lip}(\ ;\overline{\mathbb{D}})$; $=$ $(p)$. (See [9] for details.)

Remark 6.3. For values of $p$ in any compact subset of $(1;\infty)$; the corresponding extremals $f$ can all be taken to be Lip( ;

The problem of showing that the Blaschke factor has at most $N$ terms, or, even, indeed, is finite, still remains. We will discuss this more in the next section. Notice, though, that instead of Problem (6.1), we can consider the problem with $\omega$ a rational function, to get point evaluations at points other than the origin. More specifically, given $\omega(z) := \sum_{k=1}^{N} \frac{a_k}{(1-\overline{w_k}z)^2}$; $|w_k| < 1$; a linear combination of Bergman reproducing kernels, our problem becomes that of finding the extremal solutions to

$$\text{(6.7)} \qquad \sup \left\{ Re \left( \sum_{1}^{N} a_k f(w_k) \right) ; |w_k| < 1; \|f\|_{A^p} \le 1 \right\} .$$

Then the results of Khavinson and Stessin ([9]), analogous to Theorem 6.1, imply that

$$f^*(z) = CB(z) \prod_{j=1}^{2N-2} (1-\overline{\alpha_j}z)^{2/p} \prod_{1}^{N} (1-\overline{w_j}z)^{-4/p} ;$$

where $C$ is a constant, $|\alpha_j| \le 1$; $j = 1, \dots, 2N-2$; are constants and the zeros of the Blaschke product $B$ may only accumulate to those $\alpha_j$ that lie on $\mathbb{T}$.

We now turn to a discussion of what more can be said about the Blaschke factor, at least for values of $p$ close to $2$.

## 7. A continuity approach

In an attempt to shed some light on Conjecture 6.1, we begin with the following lemma.

Lemma 7.1. For

Therefore, there exists $N$ such that for $n \geq N$,

$$\mathrm{Re}\int_{\mathbb{D}} g\bar{\omega}\, dA > \mathrm{Re}\int_{\mathbb{D}} f_n\bar{\omega}\, dA + \varepsilon/2. \tag{7.1}$$

Now $g$ is continuous in the closed unit disk, and therefore for $p_n \to p$, the functions $|g|^{p_n}$ are bounded above by some constant, and therefore (by the bounded convergence theorem),

$$\gamma_n^{p_n} := \int_{\mathbb{D}} |g|^{p_n}\, dA \to \int_{\mathbb{D}} |g|^{p}\, dA = 1.$$

Note that the functions $g/\gamma_n$ have norm 1 in $A^{p_n}$.

Since $\gamma_n \to 1$ and by (7.1), there exists $M$ such that for $m \geq M$ and for $n \geq N$,

$$\frac{1}{\gamma_m}\mathrm{Re}\int_{\mathbb{D}} g\bar{\omega}\, dA > \mathrm{Re}\int_{\mathbb{D}} f_n\bar{\omega}\, dA + \varepsilon/4.$$

Choosing a large enough $n = m$ satisfying this inequality leads to a contradiction of the extremality of $f_n$.

Therefore, we must indeed have that $g = f_p$, as desired.

Note that the hypothesis in Lemma 7.2 that the functions $f_{p_n}$ converge uniformly to $f$ in the closed disk is stronger than what is really necessary for the proof: what is required is that the measures $f_{p_n} dA$ converge weakly to $f dA.$

**Corollary 7.3.** For $1 < p < \infty$, $(p)$ is a continuous function of $p$.

*Proof.* If $p_n \to p$, then, by the remark after the statement of Lemma 6.2, the functions $f_{p_n}$ are all in $\mathrm{Lip}(\ ;\overline{\mathbb{D}})$ for the some $\ ;$ and therefore form a uniformly bounded and equicontinuous family. Therefore, by the Arzela-Ascoli theorem, there exists a subsequence $f_{p_{n_k}}$ that converges uniformly in $\overline{\mathbb{D}}$ to some function $f$. By Lemma 7.2, $f = f_p.$ Therefore, by the bounded convergence theorem,

$$\int_{\mathbb{D}} f_{p_{n_k}}\bar{\omega}\, dA(z) \to \int_{\mathbb{D}} f_p\bar{\omega}\, dA(z).$$

Taking real parts, we get that $(p_{n_k}) \to (p)$. But since the function is monotone, $(p_n) \to (p)$.

**Theorem 7.4.** $\omega$ has no eros on I t Delta $> 0$ such $|p-2| <$ Delta; $f_p$ has a tN eros t in $\overline{\mathbb{D}}$.

*Proof.* If $p = 2$; then we know the solution is $f_2 = \omega$; and $\omega$ has at most $N$ zeros in the unit disk, because it is a polynomial of degree $N$.

First note that the extremal functions cannot have zeros that accumulate inside the unit disk (otherwise they would be identically zero) and therefore the zeros of the extremals can only accumulate to the boundary of the disk.

Let us first show that there exists Delta such that for $p$ in a Delta neighborhood of 2; $f_p$ has a finite number of zeros. Suppose not. Then there exists a sequence $p_n \to 2$ such that $f_{p_n}$ have infinitely many zeros in a compact neighborhood of the boundary of the disk. These zeros must have an accumulation point, and by the previous remark, this accumulation point must be on the boundary of the disk.

As in the proof of Corollary 7.3 and using Theorem 6.1, by passing to a subsequence if necessary, the $f_{p_n}$ converge uniformly in $\overline{\mathbb{D}}$ to $\omega$ by Lemma 7.2. But then

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